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Sine-Gordon Equation in Curved Space-Time

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We study the static solution of the sine-Gordon wave equation in a background geometry determined by a point mass in $1 + 1$ dimension.

The sine-Gordon system has been used in the study of a wide range of phenomena, including the propagation of crystal disturbances of waves in membranes of magnetic flux in Josephson times (Rajaraman, 1982). It also has been used in a two-dimensional model of elementary particles.

Some time back Coleman (1975; Mandelstam, 1975) established rigorously the mapping from the sine-Gordon model to the massive Thirring model (Thirring, 1958; Klaiber, 1967) by the so-called bosonization prescription in $1 + 1$ dimension. In this paper we study the static solution of the sine-Gordon wave equation (Rajaraman, 1982) in a background geometry determined by a point mass in $1 + 1$ dimension. Free-particle Klein–Gordon and Dirac equations in this kind of background have been studied by several authors (Mann *et al.*, 1991; Browr *et al.*, 1986). However, a solitary-wave-like solution for a nonlinear equation in this background has not been studied before. [For a discussion on sine-Gordon theory in flat space see Rajaraman (1982).] Our motivation for this study is twofold. One is to see, at least in the perturbation frame, what influence the background geometry has on a soliton-like solution. Also, many phenomena that are difficult to study in Fermi language have simple classical explanations in $1 + 1$ dimension in the sine-Gordon field theory.

In the following we derive the static solution of the sine-Gordon equation in the background geometry discussed in Mann *et al.* (1991). Consider two-dimensional Einstein gravity coupled to a sine-Gordon

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system. The action for this system is given by

$$S = \int d^2x\,(-g)^{1/2}\left\{R + \frac{1}{2} g^{\mu}\,\partial_\mu\phi\,\partial_\nu\phi + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\right)\phi - 1\right]\right\} \tag{1}$$

As is well known, the Hilbert–Einstein term becomes a total derivative in two-dimensional space-time and hence by suitably choosing the boundary conditions one can easily see that gravity itself has no kinetic term. Varying equation (1) in the usual way, we get the equation of motion for ϕ as

$$\frac{1}{(-g)^{1/2}}\,\partial_\mu[(-g)^{1/2} g^{\mu}\,\partial_\nu\gamma] + \frac{m^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}}{m}\,\phi\right) = 0 \tag{2}$$

Equation (2) can be simplified by the simple scaling

$$\bar{x} = mx, \qquad \bar{t} = mt, \qquad \bar{\phi} = \frac{\sqrt{\lambda}}{m}\,\phi \tag{3}$$

Then equation (2) becomes

$$\frac{1}{(-g)^{1/2}}\,\partial_\mu[(-g)^{1/2} g^{\mu}\,\partial_\nu\phi] + \sin\phi = 0 \tag{4}$$

(for the sake of brevity we omit the bar over x, t, and ϕ).

We consider equation (4) in the background gravitational field given by the metric

$$ds^2 = \alpha(x)\,dx^2 - \frac{1}{\alpha(x)}\,dx^2 \tag{5}$$

where $\alpha(x)$ satisfies the field equation

$$-\frac{d^2}{dx^2}\,\alpha(x) = 4M\,\partial(x) \tag{6}$$

We consider the following solutions for $\alpha(x)$:

$\alpha(x) = 2M|x| + 1$ corresponds to a naked source $[\alpha(x) = 2M|x| - 1$ corresponds to the (exterior) black hole solution with horizons at the Schwarzchild radius $\alpha|x| = 1/2M]$. For the solution

$$\alpha(x) = 2M|x| + 1 \tag{7}$$

equation (4) reduces to (for a static solution)

$$e^{-2Mr}\frac{d^2\phi}{dr^2} = \sin\phi \tag{8}$$

where

$$2Mr = \ln(2M|x| + 1) \tag{9}$$

The external black-hole region corresponds to the range of r, $-\infty < r < \infty$.

The static localized solution of (8) in the absence of curvature (here $M \to 0$, $r \to x$) is given by (Rajaraman, 1982)

$$\phi(r) = 4\tan^{-1}[\exp(r - r_0)] \qquad \text{soliton solution} \tag{10a}$$

$$= -4\tan^{-1}[\exp(r - r_0)] \qquad \text{antisoliton solution} \tag{10b}$$

For $M \neq 0$, the general solution of (8) cannot be obtained in closed form. However, in a weak background field the following method may be employed for a perturbative solution. We take, to the first order,

$$e^{-2Mr} \simeq 1 - 2Mr \tag{11}$$

and anticipate the solution of (8) in the form

$$\phi = \phi_0(r) + \eta(r) \tag{12}$$

where $\phi_0(r)$ is the solution given in (10a) or (10b). For definiteness we use the solution (10a).

Putting (12) in (8) and using (10a), we obtain to the first order in M

$$\frac{d^2\eta(r)}{dr^2} = \frac{8Mre^r(1 - e^{2r})}{(1 + e^{2r})^2} \tag{13}$$

The solution of (13) can be written in the form

$$\eta(r) = 8M\left[-r\tan^{-1}(e^r) - 2r\tan^{-1}(e^{-r}) - 2e^r \operatorname{Re}\Phi\left(\frac{e^r}{i}, 2, 1\right)\right] + ar + b \tag{14}$$

where

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v + n)^{-s} z^n \tag{15}$$

and a, b are integration constants. We can choose a, b such that the asymptotic behavior of ϕ for $r \to \infty$ remains unchanged, i.e., it has the same asymptotic behavior as $\phi_0(r)$. For this we find the large-r behavior of $\Phi(x/i, 2, 1)$. Noting that $\Phi(x/i, 2, 1)$ can be written as $\Phi(-x^2, 2, 1/2)$ and using the result (Gradshteyn and Ryzhik, 1987) $\Phi(z, l, v) = v^{-1}F(l, v, l + v, z)$ with $F(a, b, c, z)$ the hypergeometric function, one can get the asymptotic behavior of $e^r \operatorname{Re}\Phi(e^r/i, 2, 1)$ for the large-z behavior of $F(a, b, c, z)$.

After some algebraic steps we get

$$\eta(r) \to -4\pi rM \qquad \text{as} \quad r \to \infty$$

Hence we take $a = 4\pi M$ and $b = 0$. In Fig. 1 we show the behavior of $\phi(r)$

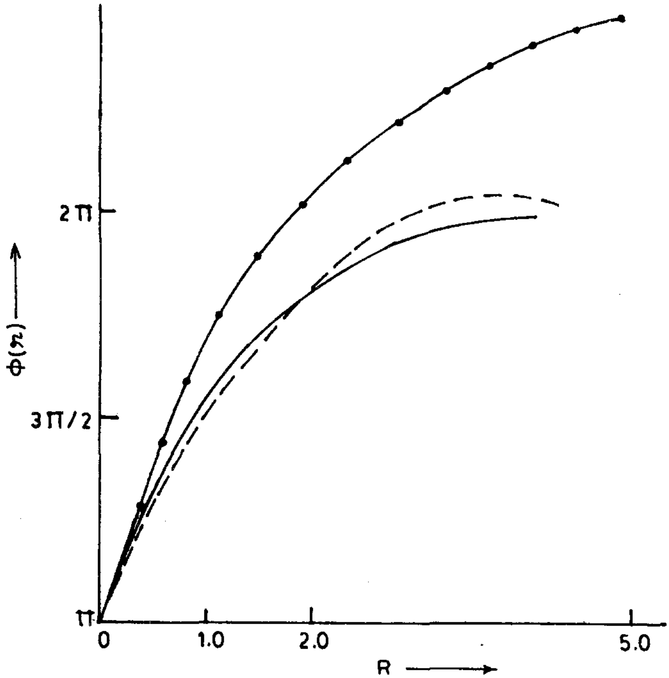

Fig. 1. Plot of $\phi(r)$ against r. (—) $\phi = 4\pi \tan^{-1}(e^r)$; (- -) present solution with $M = 0.01$; (- · -) present solution with $M = 0.1$.

in the range $0 < r < \infty$ for three values of M: $M = 0$, 0.01, and 0.1. It can be seen that for $M \neq 0$ the behavior of the graph is quite different from the $M = 0$ graph, though they merge when $r \to \infty$. But the convergence is slow if M differs significantly from zero. Our solution can be used to study the sine-Gordon field in $(1 + 1)$-dimensional curved space-time.

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